分部积分法

高等数学 I-信息、统计外招

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课程网页

分部积法

设函数 $u = u(x)$ 及 $v = v(x)$ 具有连续导数，则两个函数乘积的导数公式为

$$(uv)' = u'v + uv'$$

移项，得

$$uv' = (uv)' - u'v$$

对这个不等式求不定积分，得

$$\int uv' \mathrm{d}x = uv - \int u'v \mathrm{d}x$$

$$\int uv'\mathrm{d}x = uv - \int u'v\mathrm{d}x \qquad (\spadesuit)$$

公式($\spadesuit$)称为**分部积分公式**.

$$\int uv'\mathrm{d}x = uv - \int u'v\mathrm{d}x \qquad (\spadesuit)$$

公式($\spadesuit$)称为分部积分公式.

- 求不定积分$\int uv'\mathrm{d}x$转换为求不定积分$\int u'v\mathrm{d}x$

$$\int uv'\mathrm{d}x = uv - \int u'v\mathrm{d}x \qquad (\spadesuit)$$

公式($\spadesuit$)称为分部积分公式.

- 求不定积分$\int uv'\mathrm{d}x$转换为求不定积分$\int u'v\mathrm{d}x$
- 因为

$$\mathrm{d}u = u'\mathrm{d}x \quad \mathrm{d}v = v'\mathrm{d}x,$$

分部积分公式可以表述为

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u \qquad (\clubsuit)$$

例 1

求 $\int x\cos x\mathrm{d}x$

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解

$$\begin{aligned}\int x\cos x\mathrm{d}x &= \int x(\sin x)'\mathrm{d}x\\ &= \int x\mathrm{d}\sin x\end{aligned}$$

例 1

求 $\int x\cos x\mathrm{d}x$

解

$$\int x\cos x\mathrm{d}x = \int x(\sin x)'\mathrm{d}x$$
$$= \int x\mathrm{d}\sin x$$
$$= x\sin x - \int \sin x\mathrm{d}x$$

例 1

求 $\int x\cos x\mathrm{d}x$

解

$$\begin{aligned}\int x\cos x\mathrm{d}x &= \int x(\sin x)'\mathrm{d}x \\ &= \int x\mathrm{d}\sin x \\ &= x\sin x - \int \sin x\mathrm{d}x \\ &= x\sin x + \cos x + C\end{aligned}$$

例 2

求 $\int xe^x \mathrm{d}x$

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解

$$\begin{aligned}\int xe^x \mathrm{d}x &= \int x(e^x)\mathrm{d}x \\ &= \int x \mathrm{d}e^x\end{aligned}$$

例 2

求 $\int xe^x \mathrm{d}x$

解

$$\begin{aligned}\int xe^x \mathrm{d}x &= \int x(e^x)\mathrm{d}x \\ &= \int x\mathrm{d}e^x \\ &= xe^x - \int e^x \mathrm{d}x\end{aligned}$$

例 2

求 $\int xe^x \mathrm{d}x$

解

$$\begin{aligned}\int xe^x \mathrm{d}x &= \int x(e^x)\mathrm{d}x \\ &= \int x\mathrm{d}e^x \\ &= xe^x - \int e^x \mathrm{d}x \\ &= xe^x - e^x + C\end{aligned}$$

例 3

$\int x^2 e^x \mathrm{d}x$

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$\int x^2 e^x \mathrm{d}x$

解

$$\int x^2 e^x \mathrm{d}x = \int x^2 \mathrm{d}e^x$$

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$\int x^2 e^x \mathrm{d}x$

解

$$\begin{aligned} \int x^2 e^x \mathrm{d}x &= \int x^2 \mathrm{d}e^x \\ &= x^2 e^x - \int e^x \mathrm{d}x^2 \end{aligned}$$

例 3

$\int x^2e^x\mathrm{d}x$

解

$$\begin{aligned}\int x^2e^x\mathrm{d}x &= \int x^2\mathrm{d}e^x\\ &= x^2e^x - \int e^x\mathrm{d}x^2\\ &= x^2e^x - 2\int xe^x\mathrm{d}x\\ &= x^2e^x - 2\int x\mathrm{d}e^x\end{aligned}$$

例 3

$\int x^2 e^x \mathrm{d}x$

解

$$\begin{aligned}
\int x^2 e^x \mathrm{d}x &= \int x^2 \mathrm{d}e^x \\
&= x^2 e^x - \int e^x \mathrm{d}x^2 \\
&= x^2 e^x - 2\int x e^x \mathrm{d}x \\
&= x^2 e^x - 2\int x \mathrm{d}e^x \\
&= x^2 e^x - 2\left(x e^x - \int e^x \mathrm{d}x\right) \\
&= x^2 e^x - 2\left(x e^x - e^x\right) + C
\end{aligned}$$

例 3 说明在求不定积分过程中可以多次应用分部积分法.

- 若 u 是幂函数, 则它在分部积分法中发生降幂

$$\int x^2 e^x \mathrm{d}x \Longrightarrow \int x e^x \mathrm{d}x$$

例 4

求 $\int x \ln x \mathrm{d}x$

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解

$$\begin{aligned}\int x \ln x \mathrm{d}x &= \frac{1}{2}\int \ln x \cdot (x^2)' \mathrm{d}x \\ &= \frac{1}{2}\int \ln x \mathrm{d}x^2\end{aligned}$$

例 4

求 $\int x \ln x \mathrm{d}x$

解

$$\begin{aligned}\int x \ln x \mathrm{d}x &= \frac{1}{2}\int \ln x \cdot (x^2)' \mathrm{d}x \\ &= \frac{1}{2}\int \ln x \mathrm{d}x^2 \\ &= \frac{1}{2}\left(x^2 \ln x - \int x^2 \mathrm{d} \ln x\right)\end{aligned}$$

例 4

求 $\int x\ln x\mathrm{d}x$

解

$$\begin{aligned}\int x\ln x\mathrm{d}x &= \frac{1}{2}\int \ln x\cdot(x^2)'\mathrm{d}x \\ &= \frac{1}{2}\int \ln x\mathrm{d}x^2 \\ &= \frac{1}{2}\left(x^2\ln x - \int x^2\mathrm{d}\ln x\right) \\ &= \frac{1}{2}\left(x^2\ln x - \int x^2\frac{1}{x}\mathrm{d}x\right)\end{aligned}$$

例 4

求 $\int x\ln x\mathrm{d}x$

解

$$\begin{aligned}\int x\ln x\mathrm{d}x &= \frac{1}{2}\int \ln x\cdot(x^2)'\mathrm{d}x\\ &= \frac{1}{2}\int \ln x\mathrm{d}x^2\\ &= \frac{1}{2}\left(x^2\ln x-\int x^2\mathrm{d}\ln x\right)\\ &= \frac{1}{2}\left(x^2\ln x-\int x^2\frac{1}{x}\mathrm{d}x\right)\\ &= \frac{1}{2}\left(x^2\ln x-\int x\mathrm{d}x\right)=\frac{1}{2}\left(x^2\ln x-\frac{1}{2}x^2\right)+C\end{aligned}$$

例 5

求 $\int \arccos x \mathrm{d}x$

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求 $\int \arccos x \mathrm{d}x$

解

$$\int \arccos x \mathrm{d}x = \int \arccos x \cdot (x)' \mathrm{d}x = x \arccos x - \int x \mathrm{d} \arccos x$$
$$= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} \mathrm{d}x$$

例 5

求 $\int \arccos x \mathrm{d}x$

解

$$\begin{aligned}\int \arccos x \mathrm{d}x &= \int \arccos x \cdot (x)' \mathrm{d}x = x \arccos x - \int x \mathrm{d} \arccos x \\ &= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} \mathrm{d}x \quad \text{(分部积分法)}\end{aligned}$$

例 5

求 $\int \arccos x \mathrm{d}x$

解

$$\int \arccos x \mathrm{d}x = \int \arccos x \cdot (x)' \mathrm{d}x = x \arccos x - \int x \mathrm{d} \arccos x$$

$$= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} \mathrm{d}x \quad \text{(分部积分法)}$$

$$= x \arccos x - \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} \mathrm{d}(1-x^2)$$

例 5

求 $\int \arccos x \mathrm{d}x$

解

$$\begin{aligned}\int \arccos x \mathrm{d}x = \int \arccos x \cdot (x)' \mathrm{d}x &= x \arccos x - \int x \mathrm{d} \arccos x \\ &= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} \mathrm{d}x \quad (\text{分部积分法}) \\ &= x \arccos x - \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} \mathrm{d}(1-x^2) \quad (\text{换元积分法})\end{aligned}$$

例 5

求 $\int \arccos x \mathrm{d}x$

解

$$\int \arccos x \mathrm{d}x = \int \arccos x \cdot (x)' \mathrm{d}x = x \arccos x - \int x \mathrm{d} \arccos x$$

$$= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} \mathrm{d}x \quad \text{(分部积分法)}$$

$$= x \arccos x - \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} \mathrm{d}(1-x^2) \quad \text{(换元积分法)}$$

$$= x \arccos x - \sqrt{1-x^2} + C$$

例 6

求 $\int x \arctan x \mathrm{d}x$

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求 $\int x \arctan x \mathrm{d}x$

解

$$\int x \arctan x \mathrm{d}x = \frac{1}{2}\int \arctan x \mathrm{d}x^2$$

例 6

求 $\int x \arctan x \mathrm{d}x$

解

$$\int x \arctan x \mathrm{d}x = \frac{1}{2}\int \arctan x \mathrm{d}x^2$$
$$= \frac{1}{2}\left(x^2 \arctan x - \int x \mathrm{d} \arctan x\right)$$

例 6

求 $\int x \arctan x \mathrm{d}x$

解

$$\begin{aligned}\int x \arctan x \mathrm{d}x &= \frac{1}{2}\int \arctan x \mathrm{d}x^2 \\ &= \frac{1}{2}\left(x^2 \arctan x - \int x \mathrm{d} \arctan x\right) \\ &= \frac{1}{2}\left(x^2 \arctan x - \int \frac{x}{1+x^2} \mathrm{d}x\right)\end{aligned}$$

例 6

求 $\int x \arctan x \mathrm{d}x$

解

$$\begin{aligned}
\int x \arctan x \mathrm{d}x &= \frac{1}{2}\int \arctan x \mathrm{d}x^2 \\
&= \frac{1}{2}\left(x^2 \arctan x - \int x \mathrm{d} \arctan x\right) \\
&= \frac{1}{2}\left(x^2 \arctan x - \int \frac{x}{1+x^2} \mathrm{d}x\right) \\
&= \frac{1}{2}\left[x^2 \arctan x - \frac{1}{2}\int \frac{1}{1+x^2} \mathrm{d}(1+x^2)\right]
\end{aligned}$$

例 6

求 $\int x\arctan x\mathrm{d}x$

解

$$\begin{aligned}\int x\arctan x\mathrm{d}x &= \frac{1}{2}\int \arctan x\mathrm{d}x^2 \\ &= \frac{1}{2}\left(x^2\arctan x - \int x\mathrm{d}\arctan x\right) \\ &= \frac{1}{2}\left(x^2\arctan x - \int \frac{x}{1+x^2}\mathrm{d}x\right) \\ &= \frac{1}{2}\left[x^2\arctan x - \frac{1}{2}\int \frac{1}{1+x^2}\mathrm{d}(1+x^2)\right] \\ &= \frac{x^2\arctan x}{2} - \frac{\ln(1+x^2)}{4} + C\end{aligned}$$

例 7

求 $\int e^x \sin x \mathrm{d}x$

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解

$$\int e^x \sin x \mathrm{d}x = \int \sin x \mathrm{d}e^x = e^x \sin x - \int e^x \mathrm{d}\sin x = e^x \sin x - \int e^x \cos x \mathrm{d}x$$

例 7

求 $\int e^x \sin x \mathrm{d}x$

解

$$\int e^x \sin x \mathrm{d}x = \int \sin x \mathrm{d}e^x = e^x \sin x - \int e^x \mathrm{d}\sin x = e^x \sin x - \int e^x \cos x \mathrm{d}x$$
$$= e^x \sin x - \int \cos x \mathrm{d}e^x$$

例 7

求 $\int e^x \sin x \mathrm{d}x$

解

$$\begin{aligned}\int e^x \sin x \mathrm{d}x = \int \sin x \mathrm{d}e^x = e^x \sin x - \int e^x \mathrm{d}\sin x = e^x \sin x - \int e^x \cos x \mathrm{d}x \\ = e^x \sin x - \int \cos x \mathrm{d}e^x \\ = e^x \sin x - \left(e^x \cos x - \int e^x \mathrm{d}\cos x\right)\end{aligned}$$

例 7

求 $\int e^x \sin x \mathrm{d}x$

解

$$
\begin{aligned}
\int e^x \sin x \mathrm{d}x = \int \sin x \mathrm{d}e^x = e^x \sin x - \int e^x \mathrm{d}\sin x &= e^x \sin x - \int e^x \cos x \mathrm{d}x \\
&= e^x \sin x - \int \cos x \mathrm{d}e^x \\
&= e^x \sin x - \left(e^x \cos x - \int e^x \mathrm{d}\cos x \right) \\
&= e^x \sin x - \left(e^x \cos x + \int e^x \sin x \mathrm{d}x \right)
\end{aligned}
$$

例 7

求 $\int e^x \sin x \mathrm{d}x$

解

$$\begin{aligned}\int e^x \sin x \mathrm{d}x = \int \sin x \mathrm{d}e^x = e^x \sin x - \int e^x \mathrm{d}\sin x = e^x \sin x - \int e^x \cos x \mathrm{d}x \\ = e^x \sin x - \int \cos x \mathrm{d}e^x \\ = e^x \sin x - \left(e^x \cos x - \int e^x \mathrm{d}\cos x\right) \\ = e^x \sin x - \left(e^x \cos x + \int e^x \sin x \mathrm{d}x\right)\end{aligned}$$

解得

$$\int e^x \sin x \mathrm{d}x = \frac{1}{2} e^x (\sin x - \cos x) + C$$

例 9

求 $\int e^{\sqrt{x}}\mathrm{d}x$

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求 $\int e^{\sqrt{x}}\mathrm{d}x$

解　令 $t=\sqrt{x}$, 则 $x=t^2$, 代入所求不定积分

$$\int e^{\sqrt{x}}\mathrm{d}x=\int e^t\mathrm{d}t^2=2\int te^t\mathrm{d}t$$

例 9

求 $\int e^{\sqrt{x}}\mathrm{d}x$

解　令 $t=\sqrt{x}$, 则 $x=t^2$, 代入所求不定积分

$$\begin{aligned}\int e^{\sqrt{x}}\mathrm{d}x &= \int e^t \mathrm{d}t^2 = 2\int te^t\mathrm{d}t \quad \text{(换元积分法)} \\ &= 2\int t\mathrm{d}e^t\end{aligned}$$

例 9

求 $\int e^{\sqrt{x}}\mathrm{d}x$

解　令 $t=\sqrt{x}$, 则 $x=t^2$, 代入所求不定积分

$$\begin{aligned}
\int e^{\sqrt{x}}\mathrm{d}x = \int e^t \mathrm{d}t^2 &= 2\int te^t \mathrm{d}t \quad \text{(换元积分法)}\\
&= 2\int t\mathrm{d}e^t\\
&= 2\left(te^t - \int e^t \mathrm{d}t\right) \quad \text{(分部积分法)}
\end{aligned}$$

例 9

求 $\int e^{\sqrt{x}}\mathrm{d}x$

解　令 $t=\sqrt{x}$, 则 $x=t^2$, 代入所求不定积分

$$
\begin{aligned}
\int e^{\sqrt{x}}\mathrm{d}x = \int e^t \mathrm{d}t^2 &= 2\int te^t\mathrm{d}t \quad (\text{换元积分法}) \\
&= 2\int t\mathrm{d}e^t \\
&= 2\left(te^t - \int e^t\mathrm{d}t\right) \quad (\text{分部积分法}) \\
&= 2te^t - 2e^t + C \quad (\text{回代}t=\sqrt{x}) \\
&= 2\sqrt{x}e^x - 2e^{\sqrt{x}} + C
\end{aligned}
$$

作业

- 教材习题 4-3: 1-9; 11; 12; 16; 19; 22; 23.